Bounding the Upper Limit of Moves in the Game of Morpion Solitaire 5D

Pekka Karjalainen
May 24, 2011

A *position* in a Morpion Solitaire game is any placement of crosses and lines in a grid that is reached from the initial position by a sequence of legal moves according to the rules of the game as stated on the Morpion Solitaire web site[1]. A *move* consists of placing a new cross in the grid and drawing a line that covers four crosses already in the grid and the newly placed cross. In the initial position there are 36 crosses in the pattern specified by the rules and no lines.

The moves are played sequentially and we number them starting from one. We say that the game is *on turn N* when N moves have been played, with $N \geq 0$, and we say that a cross C is *placed* on turn N if the game is on turn N immediately after the move that includes placing C.

We define the *age* of a cross $A(C,N)$ on turn N to be the difference between the current turn N and N_C, the turn on which the cross was placed, plus one. Thus $A(C,N) = N - N_C + 1$. For completeness, we say that the initial crosses are placed in the grid on turn zero.

It follows that in any position the last cross to be placed has age one, and there is a sequence of crosses with increasing age up to the age of the initial 36 crosses, with each age lower than the maximum being represented by exactly one cross.

There are four different directions that lines may be drawn in. Since the lines cannot overlap when drawn in the same direction, any cross may be covered by at most four lines. We define the *potential* of a cross to be four minus the number of lines that cover it, and we also define the *total potential* of a position to be the sum of the potentials of all the crosses in it.

Lemma 1 *In any Morpion Solitaire 5D position reached through legal moves there must be at least six points of total potential.*

We find that the age of a cross determines the maximum number of lines that may cover it. This maximum is equal to the age of a cross with no cross being covered by more than four lines. During the first three moves the potential of the position is clearly much larger than six, so we can concentrate on positions where at least four moves have been played.

In any position the last cross to be drawn has age one, and it must have been placed during the previous move. The cross has exactly one line covering it, which was also placed during the previous move.

A cross with age two was a cross with age one before the last move, and at that time had only one line covering it. After this a single line has been added to the position, so the maximum number of lines that can cover it is two.

Similarly, a cross with age three was a cross with age two before the last move, and can have only three lines covering it in the current position.

Crosses with an age of four or higher have existed in the preceding positions long enough for them to have the maximum number of four lines covering them.

As a result, it follows that in any position there is a cross with potential of three or more, a cross with potential of two or more and a cross with potential of one or more. The sum of these limits is six.

This completes the proof of **Lemma 1**.

Since a line is always drawn to cover five crosses, adding a line reduces the total potential of a position by five. Adding a cross during the same move increases the total potential of the position by four.

It follows that each move reduces the total potential of a position by one, and the total potential of any position reached through legal moves is determined by the number of moves it took to reach the position. The initial position has 36 crosses with no lines, and thus its total potential is $36 \times 4 = 144$. When the game is on turn N, the total potential of the position is equal to $144 - N$.

By **Lemma 1**, in any legal position the total potential must be at least six. It follows that the maximum value of M, the number of moves in a game, is bounded by the inequality $144 - M \geq 6$. Solving the inequality for M, we conclude that the maximum number of legal moves in Morpion Solitaire 5D is $M \leq 138$.

Further possibilities. The result of **Lemma 1** does not take into account the relative directions of the lines drawn during the last few moves. Since two lines may only cover a single common cross when they are not drawn in the same direction and they may not overlap at all otherwise, it does not appear to be possible to construct an actual position where there are only six points of total potential remaining after a series of legal moves. This observation may lead to an improvement of the result in this proof.

References

[1] Christian Boyer. *Morpion Solitaire.* `http://www.morpionsolitaire.com/`