Paper written by Achim Flammenkamp, Germany, March 19, 2003.
PDF file created by Christian Boyer, France, February 6, 2008.

Le Morpion Solitaire

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There is a common known solitaire game in France: le Morpion. It is played on a checkered piece of paper with a pencil. The paper at the beginning contains a so called Maltesian Cross of 36 points on its grid. Your aim is to get as many as possible further points on this grid. A new point can only be set, if either a line segment of length five in one of the two diagonal directions or the two orthogonal directions could be chosen to cover this new point, provided the line segment is covered completely by existing points and at most one single point of it is already covered by other line segments.

We like to but this in a precise mathematical model: Given the grid $\mathbb{Z}^2$, then we denote by P_0 the special start configuration

$$\{(x,y) \in \mathbb{Z}^2 :\ 0 \le x,y \le 9,\ \lambda(x,y)\}$$

where λ is the characteristic boolean function of the start configuration which is defined as

$$\lambda(x,y) = \begin{cases} 3 \not| x\ , 3 \not| y : & \text{FALSE} \\ 3 \not| x\ , 3 \,|\, y\ : & 2 \,|\, \lfloor \frac{x}{3} \rfloor \ \neq\ 3 \,|\, \lfloor \frac{y}{3} \rfloor \\ 3 \,|\, x\ \ , 3 \not| y : & 3 \,|\, \lfloor \frac{x}{3} \rfloor \ \neq\ 2 \,|\, \lfloor \frac{y}{3} \rfloor \\ 3 \,|\, x\ \ , 3 \,|\, y\ : & 3 \,|\, \lfloor \frac{x}{3} \rfloor \ \vee\ 3 \,|\, \lfloor \frac{y}{3} \rfloor \end{cases}$$

using the binary boolean divide-operator $|$. Then we have $r := |P_0| = 36$. Next we define the set $\mathcal{L}$ of all line segments of length five to be

$$\{\ ((x{-}id, y{-}it))_{i\in\{-2,-1,0,1,2\}} :\ x,y \in \mathbb{Z} \quad (d,t) \in \{(1,0),(0,1),(1,-1),(1,1)\}\ \}$$

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Now we search for a as long as possible sequence of pairs of points $p_i \in \mathbb{Z}^2$ and $\ell_i \in \mathcal{L}$, — $((p_i,\ell_i))_{i\in\{1,2,\ldots,n\}}$, satisfying the following conditions:

i $p_i \notin P_0\ ,\ \forall\, j < i :\ p_i \neq p_j$

ii $p_i \in \ell_i\ ,\ \forall\, j < i :\ |\ell_i \cap \ell_j| \le 1$

iii $\exists\, i_1,i_2,i_3,i_4,i_5 \in \{1,2,\ldots,i\} :\ \ell_i = (p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$

Condition (i) means: mark each move a further point to the grid. Condition (ii) means: this point must be covered by a new line segment, but the new line segment is not allowed to have more than 1 point in common with any other one. and condition (iii) says that the new line segment must be covered completely by chosen markers.

An open problem is still to determine the maximal possible value of n. Here we like to show that this exists in deed and we give an upper bound on the maximum n.

For this we define for every $k \leq n$ an evaluation function f_k on $\mathbb{Z}^2$ depending on the set $P_i := P_0 \cup \{p_i\}_{i\in\{1,2,\ldots,k\}}$ and the set $\{\ell_i\}_{i\in\{1,2,\ldots,k\}}$ of the points and line segments in our sequence. To do this we need the $\forall \ell \in \mathcal{L}$ the abbreviation of

$$\ell^* := (p_2, p_3, p_4) \quad \text{if} \quad \ell = (p_1, p_2, p_3, p_4, p_5)$$

which is simple the middle part consisting of the inner 3 points of a line segment.

Now we can define for every $k \leq n$ and each $((p_i, \ell_i))_{i\in\{1,2,\ldots,n\}}$

$$f_k(q) \;=\; 8\delta_{q\in P_k} - \sum_{i\in\{1,2,\ldots,k\}} \delta_{q\in\ell_i} - \sum_{i\in\{1,2,\ldots,k\}} \delta_{q\in\ell_i^*}$$

This has the implications that always holds

$$\forall\, q \in \mathbb{Z}^2 : \quad 0 \leq f_k(q) \leq 8$$

with the lower bound is due to the fact that the line segments have 4 possible directions and for each direction there is either at most one full cover by the middle of a line segment or two half covers by line ends. Moreover, the fact that each further point p_i adds 8 and each further line ℓ_i decreases by $5 + 3$ its covered points, we get — summing up over the whole plane —

$$\forall\, k : \quad \sum_{q\in\mathbb{Z}^2} f_k(q) = 8r$$

which is an invariante for each k and any sequence.

A point $q \in P_k$ for whichs holds $f(q) = 0$ must be saturated in all 4 directions and must be an inner point of the set P_k. Thus the points with $f > 0$ must contain at least the margin points of the set because these can't be saturated completed from all 4 directions.

Therefore the set $P_k \subset \mathbb{Z}^2$ should be have a border measure as low as possible. This leads immediately to a simple upper bound: Giving $8r$-many border points for a subset S the maximization of $|S|$ means S must be at least a convex set and because it is a subset in $\mathbb{Z}^2$ it must be a convex octagon. Lucky an octagon has 8 sides, so the full-symmetric octagon with side length $r + 1$ has exactly $8r$ border points and a total number of $7r^2 + 4r + 1$ points.

This rough estimation can be improved by refinement, still assuming we can cover the inner points of the optimal convex octagon perfectly, such that only near the margin of the octagon $f > 0$. In detail look at a typical corner of such an octagon:

$$\begin{array}{ccccccc}
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & 4 & 3 & 3 & 3 \\
. & . & 3 & 1 & 0 & 0 & 0 \\
. & 3 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Here a dot . indicates a point $q \notin P$, having $f(q) = 0$, else the value $f(p_i)$ is shown. Assuming the horizontal and vertical diameters of the octagon to be b_1 and b_2 and the four corner cut-off-sizes to be $a_1, a_2, a_3, a_4 \leq \min\{b_1, b_2\}$, we get the condition

$$3(2b_1 + 2b_2) - 2(a_1 + a_2 + a_3 + a_4) - 4 \leq 8r \tag{1}$$

for the border weight.

Under this condition we have to maximize

$$b_1 b_2 - \sum_{j \in \{1,2,3,4\}} \frac{a_j(a_j + 1)}{2} \tag{2}$$

the total number of points in respect to $b_1, b_2, a_1, a_2, a_3, a_4$. Without loss of generality we may assume that $b_1 \leq b_2$ and $a_1 \leq a_2 \leq a_3 \leq a_4$. Because of the special form of the restriction and the objective function we know that for the maximum must hold $b_1 \leq b_2 \leq b_1 + 1$ and $a_1 \leq a_4 \leq a_1 + 1$. So, renaming a_1 to a and b_1 to b, we rewrite the restriction to

$$12b + 6\beta - 8a - 2\sigma - 4 \leq 8r$$

with $b_2 - b_1 = \beta \in \{0, 1\}$ and $\sum_j a_j - a = \sigma \in \{0, 1, 2, 3\}$ and have to maximize

$$b^2 + \beta b - (a + 1)(2a + \sigma)$$

in respect to b, β, a, σ. Because σ and a can be used to get equality in the restriction still keeping the same maximum, we get with this "active" restriction

$$a = a(b, \beta) = \lfloor \frac{6b + 3\beta - 2 - 4r}{4} \rfloor \geq 0$$

and

$$\sigma = 6b + 3\beta - 2 - 4r - 4a \quad .$$

Substituting a and σ in the objective function, keeping in mind that $6b + 3\beta \geq 4r + 2$ must hold, we yield

$$\max_{b,\beta} \quad b^2 + \beta b - (a(b, \beta) + 1)(6b + 3\beta - 4r - 2 - 2a(b, \beta))$$

For $r = 36$, the maximum value 741 is attained with $b = 31$ and $\beta = 0$, implying $a = 10$ and $\sigma = 0$.

Remark: If we would drop the requirement that a_1, a_2, a_3, a_4 and b_1 and b_2 must be integers, we could approximate the solution by knowing that in the maximum must hold $a_1 = a_2 = a_3 = a_4$, $b_1 = b_2$ and the restriction $6(2b) - 8a - 4 \leq 8r$ must hold with equality. Putting this together, we have to maximize the term $b^2 - \frac{1}{2}((3b - 2r)^2 - 1)$ in b, which means that $2b - (3b - 2r)3 = 0$ must hold. Then we would get for the optimal b the value $\frac{6r}{7}$ and the maximal value $\frac{4r^2}{7} + \frac{1}{2}$, resulting in the case $r = 36$ in a maximal value of $741.0714\ldots$ as good as our bound on n in the exact all integer estimation.

To get a better bound we like to estimate the size of P_n better. The main idea is to use the alignment restriction of size 5. In our model each points has costs 8 and therefore our aim is to cover as much as possible points with lines in all four directions to push this down to 0. Denote the horizontal diameter, this is the number of lines orthogonal to the x-axis, again by b_1 and the vertical diameter, the number of parallel lines to the x-axis, with b_2. Furthermore let the diameter in the two diagonal directions be d_1 and d_2. Then we get a to (2) equivalent formula for the number of points covered in all 4 directions to be

$$b_1 b_2 - \lfloor \frac{(b_1 + b_2 - d_1)^2}{4} \rfloor - \lfloor \frac{(b_1 + b_2 - d_2)^2}{4} \rfloor \tag{3}$$

Because the two end points of a line of length $4k+5$ are only covered half, we get a waste of 2 each such line. Furthermore because of the alignment restriction mod4, we must have an average access of $1/4 \sum_{i=0}^{3} i$ each independent line direction. Because of the 4 directions exactly 2 are independend, we get at least for the lines in the other 2 directions this costs for the miss-aligment. Each point in this access, which can't be covered gives a further costs of 2. Trying to avoid mis-alignment at the first and last lines of each line-bundle can save at most 3/2 for each, but on the other hand we have to pay $2 \cdot 2$ due to the fact that no line of length 1 is availible for each such bundle which out weights this small win – so we neglect this small effect. Because $d_1, d_2 \geq \max\{b_1, b_2\}$ we can put all together to the condition

$$2(b_1 + b_2 + d_1 + d_2) + 2(\alpha b_1 + (3 - \alpha) b_2) \leq 8r \qquad 0 \leq \alpha \leq 3 \tag{4}$$

Maximizing (3) under the asssumption $b_1 \leq b_2$, results in the non-integer case in the unique solution

$$b_1 = \frac{3}{8}r \quad b_2 = \frac{6}{8}r \quad d_1 = d_2 = \frac{7}{8}r \quad \alpha = 3$$

which give the maximal value of at most $\frac{r^2+2}{4}$. The all integer solution for our case $r = 36$, give a maximal value of 324 with the two different solutions $b_1 = 14, b_2 = 26, d_1 = d_2 = 31, \alpha = 3$ and $b_1 = 13, b_2 = 28, d_1 = d_2 = 32, \alpha = 3$ — for each solution (4) is active.